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Araştırma Makalesi / Research Article

Seyrek Cesàro Matrisleri

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Anahtar kelimeler
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Özet

Bu çalışmada, serilerin üst yakınsaklığı ile bu serilerin kısmi toplamlar dizisinin toplanabilir bir uzanımının varlığı arasında bağlantı kuran bir genel toplanabilme matrisi tanıtıldı. Bu matrislerin bazı özellikleri ve Riesz Matrislerinin hangi koşullar altında bu matris sınıfına girdiği incelendi.

On Diluted Cesàro Matrices

Keywords
Elongation, CesàroMeans, Riesz Matrix

Abstract

In this paper, we introduced some general summability matrices which make contact between the overconvergence of series and the existence of a summable elongation of the sequence of the partial sums of the series. We investigated some properties of them and analysed under what conditions will the Riesz Matrices be in the class of matrices which are defined.

1. Introduction

A theorem of Drobot (1970) asserts that a power series can be diluted to become C_1-summable outside of the circle of convergence if and only if it overconvergences, that is, there exists a convergent subsequence of the sequence of partial sums of power series outside of the circle of convergence. Gharibyan and Luh (2011) obtained the same result by the different method. The authors in (Tunc and Kucukaslan, 2014) extended this result to some regular summability matrices instead of C_1-summability. The natural question is what are general summability matrices that replace with Cesàro Matrices in the theorem. This paper deals with this problem. We define some general summability matrices which are called as "diluted Cesàro matrices", investigate some properties and analysed under what conditions will the certain important methods of summability be diluted Cesàro matrices.

2. Material and Method

We give some required concepts in the following.

2.1. Elongation of Sequences

Let

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots \tag{2.1}$$

be a series of complex or real numbers of a_n, $n \in \mathbb{N}$. The series

$$\sum_{n=1}^{\infty} a_n' = a_1 + 0 + 0 + \cdots 0 + a_2 + 0 + 0 + \cdots 0 + a_3 + \cdots$$

which is obtained by adding arbitrary number of zeros between the terms of the series (2.1) is called a “dilution” of the series (2.1). The number of zeros which are added can be characterized by a sequence of natural numbers. Let $m_n - 1$, $n \in \mathbb{N}$, be the number of zeros added after the term a_n, and let $m=(m_n)_{n=1}^{\infty}$. Then the resulting series is called $m-$dilution of the series(2.1). In the special case, if m is a constant sequence, it is said that this dilution is uniform. If $(S_n)_{n=1}^{\infty}$ is the sequence of partial sums of the series (2.1), then the sequence of partial sums of the diluted series will be in the following form:

$$(\underbrace{S_1,S_1,...,S_1}_{m_1-times},\underbrace{S_2,S_2,...,S_2}_{m_2-times},...,\underbrace{S_n,S_n,...,S_n}_{m_n-times},...) \quad (2.2)$$

This result leads to the concept of elongation of the sequences. The sequence (2.2) is called $m-elongation$ of (S_n). It is obvious that the sequence (S_n) is convergent if and only if any $m-$ elongation of (S_n) is convergent with the same limit. However, this assertion does not hold for the Cesàro convergence of sequences. For example, the Cesàro limit of the sequence $(x_n)=(1,0,1,0,\cdots)$ is $\frac{1}{2}$, while the Cesàro limit of $m=(1,2,1,2,1,2,\cdots)$ -elongation of the sequence (x_n) is $\frac{1}{3}$.

2.2. A-Summability

Let $A=(a_{n,k})$, $n,k\in\mathbb{N}$, be an infinite matrix of real (or complex) numbers. A sequence (S_n) of real (or complex) numbers is said to be summable to a number S by the method $A=(a_{n,k})$, shortly A-summable to S, if the limit relation

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}a_{n,k}S_k=S$$

holds, and it is written as $\lim_A S_n=S$. The matrix $A=(a_{n,k})$ is called regular if it transforms convergent sequences to convergent sequences with the same limit.

The Cesàro Matrix: The Cesàro means C_1 which transform a given sequence (S_n) to the sequence (σ_n), where

$$\sigma_n=\frac{1}{n}\sum_{k=1}^{n}S_k,$$

which has the matrix representation $(a_{n,k})$ defined by

$$a_{n,k}=\begin{cases}\frac{1}{n}, & k\le n\\ 0, & k>n.\end{cases}$$

The matrix $C_1=(a_{n,k})$ is regular (Petersen, 1969).

Riesz Matrices: Suppose that (p_n) is a sequence of non-negative numbers, and put

$$P_n=p_1+p_2+...+p_n; \qquad p_1>0.$$

Thematrix transformation of (S_n) given by

$$R_n=\frac{1}{P_n}\sum_{k=1}^{n}p_kS_k$$

is called as the Riesz mean (R,p_n). The matrix representation of the (R,p_n) mean is defined as follows

$$r_{n,k}=\begin{cases}\frac{p_k}{P_n}, & k\le n\\ 0, & k>n.\end{cases}$$

If we take $p_n=1$ for all $n\in\mathbb{N}$, then (R,p_n) coincide with the Cesàro means C_1. Besides, (R,p_n) is regular if and only if $\lim_{n\to\infty}P_n=\infty$ (Petersen, 1969).

3. Diluted Cesàro Matrices

Let (x_n) be an arbitrary sequence. While the C_1 transformation of an elongation of (x_n) is convergent, the sequence (x_n) may be not C_1-convergent. For example, the sequence

$$(x_n)=(\overset{2^0}{1},\overset{2^0}{0},\ \overset{2^1-\text{times}}{1,1},\ \overset{2^1-\text{times}}{0,0},\ \overbrace{1,1,1,1}^{2^2-\text{times}},\overbrace{0,0,0,0}^{2^2-\text{times}},\cdots)$$

is not C_1-convergent(Peyerimhoff, 1969), but it has an elongation whose Cesàro means converge, since it is bounded (Drobot, 1975). The following theorem is concerned with the conditions that if the sequence has a C_1-convergent elongation then it is C_1-convergent.

First, we give a lemma, which is obtained by simple calculations.

Lemma 3. 1. *Let* $m=(m_n)$ *be a sequence of positive integers and* $M_n=\sum_{k=1}^{n} m_k$ *. Then*

$$\sum_{k=1}^{n-1} M_k\left(\frac{1}{m_k}-\frac{1}{m_{k+1}}\right)=n-\frac{M_n}{m_n}.$$

Theorem 3. 1. *Let* (x_n) *be a sequence of real numbers,* $m=(m_n)$ *be a sequence of positive integers and* $M_n=\sum_{k=1}^{n} m_k$ *. If m-elongation of* (x_n) *is* C_1*-convergent to a number* ℓ *, and the conditions*

i. $\lim_{n\to\infty}\frac{M_n}{nm_n}=\alpha<\infty$

ii. $\sup_n \frac{1}{n}\sum_{k=1}^{n-1} M_k\left|\frac{1}{m_k}-\frac{1}{m_{k+1}}\right|<\infty$

hold, then (x_n) *is* C_1*-convergent to* ℓ *.*

Proof. Let $(\overline{x}_n)$ be m-elongation of the sequence (x_n) and $\overline{\sigma}_n$ be the Cesàro means of the elongated sequence. Then the sequence $(\overline{\sigma}_n)$ converges to the number ℓ . Since every subsequence of $(\overline{\sigma}_n)$ converges to the number ℓ , then

$$\lim_{k\to\infty}\overline{\sigma}_{M_k}=\lim_{k\to\infty}\frac{1}{M_k}\sum_{i=1}^{k} m_i x_i=\ell.$$

Let σ_n be the Cesàro means of the sequence (x_n). Using Abel's partial summation formula, we have

$$\sigma_k=\frac{1}{k}\sum_{i=1}^{k} x_i=\frac{1}{k}\sum_{i=1}^{k}\frac{1}{m_i} m_i x_i$$

$$=\frac{1}{k}\left\{\frac{1}{m_k}\sum_{i=1}^{k} m_i x_i+\sum_{i=1}^{k-1}\left(\frac{1}{m_i}-\frac{1}{m_{i+1}}\right)\sum_{j=1}^{i} m_j x_j\right\}$$

$$=\frac{M_k}{km_k}\overline{\sigma}_{M_k}+\sum_{i=1}^{k-1}\frac{M_i}{k}\left(\frac{1}{m_i}-\frac{1}{m_{i+1}}\right)\overline{\sigma}_{M_i}.$$

By the hypotheses of theorem and using Lemma 1, the matrix $A=(a_{ki})$, with entries

$$a_{ki}=\begin{cases}\frac{M_i}{k}\left(\frac{1}{m_i}-\frac{1}{m_{i+1}}\right), & i<k\\ 0, & i\geq k\end{cases}$$

satisfies the conditions of Kojima-Schur Theorem (Cooke, 1950). Therefore, we obtain

$$\lim_{k\to\infty}\sigma_k=\alpha\ell+(1-\alpha)\ell=\ell. \blacksquare$$

We will denote by ω the space of all sequences with real terms. Besides the space ω, let us consider the subspaces of ω which are given in the following:

$$l_\infty:=\{(x_n)\in\omega:\ \sup_n|x_n|<\infty\}$$

$$c:=\{(x_n)\in\omega:\ \lim_{n\to\infty} x_n \text{ exists}\}$$

$$c_0:=\{(x_n)\in\omega:\ \lim_{n\to\infty} x_n=0\}$$

It is clear that $c_0\subset c\subset l_\infty\subset\omega$. Let N be a non-empty subset of ω.

Definition 3. 1.Let $A=(a_{nk})$ be a limitation matrix which transforms convergent sequences to convergent sequences. The matrix A is called a Diluted Cesàro Matrix on N (for short DCM_N) if every A-convergent sequence on N has C_1-convergent elongation. If $N=l_\infty$, it is called as Diluted Cesàro Matrix (for short DCM).

For example, every Schur Matrix is a DCM , since it transforms each bounded sequence to a convergent sequence.

Definition 3. 2. Let $A=(a_{nk})$ be a limitation matrix which transforms convergent sequences to convergent sequences. The matrix A is called a Smooth Diluted Cesàro Matrix on N (for short $SDCM_N$) if every A-convergent sequence on N has C_1-convergent elongation with same limit. If $N=l_\infty$, it is called as Smooth Diluted Cesàro Matrix (for short $SDCM$).

It is obvious that every $SDCM_N$ is a DCM_N. But inverse of this statement is not true. For example the matrix $A=(a_{nk})$ with entries

$$a_{nk}=\begin{cases}1/2, & k=4^n\\ 1/2, & k=4^n-1\\ 0, & \text{others}\end{cases}$$

is a regular matrix and a DCM, but is not a $SDCM$. Because, for the sequence

$$(x_n)=(\overset{2^0}{1},\ \overset{2^1\text{ times}}{0,0}\ ,\overbrace{1,1,1,1}^{2^2\text{ times}},\overbrace{0,0,...,0}^{2^3\text{ times}},\overbrace{1,1,...,1}^{2^4\text{ times}},0,\cdots)$$

we have

$$\lim_A x_n=\frac{1}{2},$$

but the sequence (x_n) is not elongated as being C_1-convergent.

Proposition 3. 1. *Let $N\subset M\subset\omega$. If a matrix A is a DCM_M, then it is a DCM_N.*

Remark 3. 1. The converse of this proposition is not true. Indeed, if we consider the matrix $A=(a_{nk})$ with entries

$$a_{nk}=\begin{cases}-1/2, & k=n\\ 1/2, & k=n+1\\ 0, & \text{others,}\end{cases}$$

it is clear that the matrix A is a $SDCM_{c_0}$, but it is not a $SDCM_{l_\infty}$. Moreover, it is a DCM_{c_0}, but it is not a DCM_ω, since the sequence $(x_n)=(n)$ does not have a C_1-convergent elongation, but $\lim_A x_n=1/2$.

Theorem 3. 2. *If $c\subset N\subset\omega$, then every $SDCM_N$ is a regular matrix transformation.*

Proof. Let A be a $SDCM_N$ and (x_n) be a convergent sequence with limit ℓ. Then every elongation of the sequence is C_1-convergent to ℓ. Therefore, $\lim_A x_n=\ell$. ■

Remark 3. 2. There exists a $SDCM_N$ which is not regular matrix transformation. For an example, the matrix A in Remark 1 is a $SDCM_{c_0}$ but it is not regular, since it transforms every convergent sequence to a sequence converging zero.

The identity matrix I, the Zweier matrix $Z_{1/2}$ and the Cesàro matrix (of order 1) C_1 are all $SDCM_\omega$. If (p_n) be an increasing sequence of non-negative real numbers, then the Riesz matrix (R,p_n) is a $SDCM$, since C_1 is stronger than (R,p_n) in this case. But, there exists a Riesz matrix (R,p_n) which is not a $SDCM$. For example, it is clear that the Riesz matrix (R,p_n) with

$$p_k=\begin{cases}1, & k=1\\ 1, & k=n+\sum_{i=0}^{n-1}2^i \quad\text{or}\quad k=n+1+\sum_{i=0}^{n-1}2^i\\ 0, & \text{others}\end{cases}$$

Where $n\in\mathbb{N}$, is a regular matrix. However, it is not a $SDCM$, since the (R,p_n)-limit of the sequence

$$(x_n)=(\overset{2^0}{1,0},\overset{2^1}{1,0,0},1,\overbrace{0,0,0,0}^{2^2-\text{times}},1,\overbrace{0,0,...,0}^{2^3-\text{times}}1,0\cdots)$$

is $1/2$. But there is no an elongation of this sequence which is C_1-convergent to $1/2$, by the Theorem 2 in (Drobot, 1975).

The following theorems are concerned with the problem: Under what conditions will the Riesz matrix (R,p_n) be a $SDCM$?

Theorem 3. 3. *Let (p_n) be a decreasing sequence of non-negative real numbers such that*

i. $\lim_{n\to\infty}p_n=p\in\mathbb{N}$,
ii. The sequence $(n(p_n-[\![p_n]\!]))$ is bounded.

Then the Riesz matrix (R,p_n) is a $SDCM$.

Proof. By the condition *i*, the Riesz matrix (R,p_n) is regular. Let $(x_n)\in l_\infty$, that is for a number $M>0$, $|x_n|\le M$ for all $n\in\mathbb{N}$. Moreover, let

$$\lim_{(R,p_n)}x_n=\lim_{n\to\infty}\frac{1}{P_n}\sum_{k=1}^{n}p_kx_k=\alpha \tag{3.1}$$

where $P_n=p_1+p_2+\cdots+p_n$. For $m_n=[\![p_n]\!]$, $n\in\mathbb{N}$, let the $m=(m_n)$-elongation of the sequence (x_n) be given by

$$(y_n)=(\overbrace{x_1,x_1,\cdots,x_1}^{m_1 \text{ times}},\overbrace{x_2,\cdots,x_2}^{m_2 \text{ times}},\cdots,\overbrace{x_k,\cdots,x_k}^{m_k \text{ times}},\cdots)$$

For the proof, we have to prove the following:

$$\lim_{C_1} y_n=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}y_k=\alpha$$

By the condition i, there exists $r\in\mathbb{N}$ such that $m_n=p$ for all $n\geq r$. Any natural number $n\geq r$ has a representation of the form

$$n=\sum_{k=1}^{r-1}m_k+ip+s,\quad i\in\mathbb{N},\ 0\leq s<p,$$

$\sum_{k=1}^{0}:=0$. Then, we get

$$\tau_n=\frac{1}{n}\sum_{k=1}^{n}y_k=\frac{1}{n}\left(\sum_{k=1}^{r+i-1}m_kx_k+sx_{N+i}\right)$$

$$=\frac{P_{r+i-1}}{n}\left(\frac{1}{P_{r+i-1}}\sum_{k=1}^{r+i-1}p_kx_k\right)-$$

$$\frac{1}{n}\sum_{k=1}^{r+i-1}(p_n-[\![p_n]\!]))\,x_k+\frac{sx_{r+i}}{n}$$

By the condition *ii*, we have

$$\lim_{i\to\infty}\frac{P_{r+i-1}}{n}=\lim_{i\to\infty}\frac{P_{r+i-1}}{\sum_{k=1}^{r-1}m_k+ip+s}$$

$$=\lim_{i\to\infty}\frac{(r+i-1)p+\log(r+i-1)}{\sum_{k=1}^{r-1}m_k+ip+s}=1$$

Thus by (3.1), we obtain

$$\lim_{i\to\infty}\frac{P_{r+i-1}}{n}\left(\frac{1}{P_{r+i-1}}\sum_{k=0}^{r+i-1}p_kx_k\right)=\alpha.$$

On the other hand

$$0\leq\lim_{i\to\infty}\frac{1}{n}\sum_{k=1}^{r+i-1}|\,p_n-[\![p_n]\!]\,|\,|x_k|$$

$$\leq M\lim_{i\to\infty}\frac{\log(r+i-1)}{\sum_{k=0}^{r-1}m_k+ip+s}=0$$

and since

$$\lim_{i\to\infty}\frac{sx_{r+i}}{n}=0$$

we have

$$\lim_{n\to\infty}\tau_n=\alpha\ .\blacksquare$$

Note that if $p=0$, then Theorem 3. 3 does not apply to decide a Riesz matrix is a $SDCM$, or not. We give the following theorem for this situtation, but first, we prove a lemma.

Lemma 3. 2. *If the sequence (p_n) of positive numbers is decreasing and the sequence $\left(\frac{P_n}{np_n}\right)$ is bounded, then the sequence $\left(\frac{1}{n}\sum_{k=1}^{n-1}\frac{p_k-p_{k+1}}{p_kp_{k+1}}P_k\right)$ is bounded too, where $P_n=p_1+p_2+\cdots+p_n$.*

Proof. Applying Abel's partial summing formula, we get

$$\sum_{k=1}^{n-1}\frac{p_k-p_{k+1}}{p_kp_{k+1}}P_{k+1}=$$

$$=P_n\sum_{k=1}^{n-1}\left(\frac{1}{p_{k+1}}-\frac{1}{p_k}\right)-\sum_{k=1}^{n-2}(P_{k+2}-P_{k+1})\sum_{i=1}^{k}\left(\frac{1}{p_{i+1}}-\frac{1}{p_i}\right)$$

$$=P_n\left(\frac{1}{p_n}-\frac{1}{p_1}\right)-\sum_{k=1}^{n-2}p_{k+2}\left(\frac{1}{p_{k+1}}-\frac{1}{p_1}\right)$$

$$=\frac{P_n}{p_n}-\frac{P_n}{p_1}-\sum_{k=1}^{n-2}p_{k+2}\left(\frac{1}{p_{k+1}}-\frac{1}{p_1}\right)\leq\frac{P_n}{p_n}$$

Since

$$\sum_{k=1}^{n-1}\frac{p_k-p_{k+1}}{p_kp_{k+1}}P_k\leq\sum_{k=1}^{n-1}\frac{p_k-p_{k+1}}{p_kp_{k+1}}P_{k+1},$$

we obtain desired result.■

Theorem 3. 4. *Let (p_n) be a decreasing sequence of positive real numbers such that*

i. $\lim_{n\to\infty}p_n=0$.

ii. $\sup_n \left(P_n / \left(np_n \right) \right) < \infty$

Then the Riesz matrix $\left(R, p_n \right)$ *is a* $SDCM$.

Proof. The proof is clear by Lemma 2 and Theorem 1.4.7 in (Petersen, 1966). ■

3.1. An Application

Let us consider a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{with} \quad \limsup_{n\to\infty} \left| a_n \right|^{\frac{1}{n}} = 1 \qquad (3.2)$$

and denote its partial sums by

$$S_n(z) = \sum_{k=0}^{n} a_k z^k \qquad (3.3)$$

It is well known that the sequence of the partial sums $\left(S_n \right)_{n=0}^{\infty}$ is uniformly convergent to the function f on each compact subset of the unit disk $\mathrm{D} := \left\{ z : \left| z \right| < 1 \right\}$ and divergent for all z, where $\left| z \right| > 1$. It is also known that it can be constructed such a power series with the property that a certain subsequence of $\left(S_n \right)$ $\{S_n\}$ converges to f on the open sets from exterior of the unit disk where the function f is regular (Porter, 1906-1907). This is the phenomenon of overconvergence.

A power series in (3.2) is called overconvergent, if there exist an open set $U \subset \left\{ z : \left| z \right| \geq 1 \right\}$ and a monotone increasing sequence of positive integers $\left\{ n_k \right\}$ such that $\left(S_{n_k} \right)$ converges compactly on U.

We may give the result of Gharibyan and Luh (2011) in the following form.

Theorem 3. 5. *Let* A *be a* $SDCM$ *and (3.2) has an analytic continuation. Then the sequence* $\left(S_n \right)$ *given by (3.3) is A-convergent compactly in an open set U outside the unit disk then the power series is overconvergent.*

4. Conclusions

The class of Diluted Cesàro Matrices which is a new class of summability matrices, generalizes the the condition of overconvergence of series with Dirichlet type. It is important to determine the charactheristic properties of these matrices forthe theory of divergent series. In this study we generalized the results of Gharibyan, and Luh (2011), Drobot (1970), Luh and Nieß (2013).

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